Conformal Nets,
Maximal Temperature
and Models from Free Probability

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Abstract

We consider conformal nets on S^1 of von Neumann algebras, acting on the full Fock space, arising in free probability. These models are twisted local, but non-local. We extend to the non-local case the general analysis of the modular structure. The local algebras turn out to be III_1-factors associated with free groups. We use our set up to show examples exhibiting arbitrarily large maximal temperatures, but failing to satisfy the split property, then clarifying the relation between the latter property and the trace class conditions on $e^{-\beta L}$, where L is the conformal Hamiltonian.

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1 Introduction

This note grew up as an attempt to combine ideas from Free Probability [20] and Algebraic Quantum Field Theory [15].

In this spirit, we construct conformal nets on S^1 of von Neumann algebras acting on the full Fock space, generalizing the single von Neumann algebra construction in Free probability [19]. Such models arise by second quantization with Boltzmann statistics of the one-particle Hilbert spaces associated with derivatives of the $U(1)$-current algebra. The local algebras are III_1 factors, but not approximately finite-dimensional, indeed they have cores isomorphic to $L(\mathbb{F}_\infty) \otimes B(\mathcal{H})$, where $L(\mathbb{F}_\infty)$ is the von Neumann algebra generated by the left regular representation of the free group $\mathbb{F}_\infty$ on infinitely many generators [18, 19].

These nets are not local, but satisfy twisted locality, therefore we are led to extend to the non-local case the general analysis of conformal nets on S^1, in particular concerning the geometric description of the modular structure, cf. [6, 12].

As a consequence, we clarify the relations between the trace class property for the semigroup generated by the conformal Hamiltonian and the split property, i.e. the statistical independence of the observable von Neumann algebras associated to disjoint intervals with positive distance.

More specifically, let $\mathcal{A}$ be a conformal net on S^1 with conformal Hamiltonian L. As is known, if $\mathcal{A}$ satisfies the trace class condition at all $\beta > 0$ with $\mathrm{Tr}(e^{-\beta L}) \leq e^{a\beta^{-r}}$ for some constants $a > 0$ and $r > 0$, then $\mathcal{A}$ is split [9].

Here we point out first that no requirement on the growth of $\mathrm{Tr}(e^{-\beta L})$ is necessary, namely the trace class condition $\mathrm{Tr}(e^{-\beta L}) < 0$, $\forall \beta > 0$, alone implies the split property, and second that, on the other hand, the trace class condition at one fixed $\beta > 0$ is not sufficient.

The first statement is a rather direct consequence of results in [8]. Concerning the second one, we construct conformal nets on S^1 that violate the split property, although a trace class condition for $e^{-\beta L}$ is satisfied with a maximal temperature β_0^{-1}, namely

$$\mathrm{Tr}(e^{-\beta L}) < \infty \Leftrightarrow \beta > \beta_0,$$

where β_0 may be arbitrarily small.

So far the Operator Algebras analysis of Conformal Quantum Field Theory has been mostly restricted to the local case, namely to the "observable algebra" case. However non-local nets appear naturally, for instance in the Fermionic case.

We thus take this opportunity to develop the general analysis somewhat in more detail than necessary. Some of these results have the same proof as in the local case and are mentioned for completeness. Other results, as modular covariance, need however an adaptation.

2 General properties of conformal nets on $\mathbf{S}^1$

By an interval I of S^1 we shall always mean a *proper* interval, namely I and its complement I' are assumed to have non-empty interiors. We denote by $\mathcal{I}$ the set of

intervals of S^1.

A *conformal precosheaf* (or *conformal net*[1]) $\mathcal{A}$ of von Neumann algebras on the intervals of S^1 is a map

$$I \to \mathcal{A}(I)$$

from $\mathcal{I}$ to the set of von Neumann algebras on a Hilbert space $\mathcal{H}$ which verifies the following properties 1,2,3,4:

1. ISOTONY : *If I_1, I_2 are intervals and $I_1 \subset I_2$, then*

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \ .$$

In the following the Möbius group $PSL(2,\mathbb{R})$, hence its universal covering G, acts as usual by diffeomorphisms of S^1.

2. CONFORMAL INVARIANCE: *There is a representation U of G on $\mathcal{H}$ such that*

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \ , \qquad g \in G, \ I \in \mathcal{I}.$$

3. POSITIVITY OF THE ENERGY : *The generator of the rotation subgroup $\vartheta \to U(R(\vartheta))$ is positive*, where $R(\vartheta)$ denotes the (lifting to G of the) rotation by an angle ϑ (in the following we shall often write $U(\vartheta)$ instead of $U(R(\vartheta))$).

Let I_0 be the upper semi-circle. We identify as usual I_0 with the positive real line $\mathbb{R}_+$ via the Cayley transform and we consider the one parameter groups $\Lambda_{I_0}(t)$ and $T_{I_0}(t)$ of diffeomorphisms of S^1 that are conjugate by the Cayley transform respectively to the dilations $x \to e^t x$ and translations $x \to x + t$ on $\mathbb{R}$. Moreover we consider the reflection of S^1 given by $r_{I_0} : z \to \bar{z}$ where $\bar{z}$ is the complex conjugate of z.

For a general $I \in \mathcal{I}$ we choose $g \in G$ such that $I = gI_0$ and set

$$\Lambda_I = g\Lambda_{I_0}g^{-1}, \quad r_I = gr_{I_0}g^{-1}, \quad T_I = gT_{I_0}g^{-1} \ .$$

(T_I is however well-defined only up to a rescaling of the parameter).

Recall the equivalence between the positivity of the conformal Hamiltonian and the positivity of the usual Hamiltonian energy, see e.g. Lemma B.5 in [13].

4. EXISTENCE OF THE VACUUM: *There exists a unit U-invariant vector Ω (vacuum vector) which is cyclic for the von Neumann algebra $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$ and separating for the von Neumann algebra $\cap_{I\in\mathcal{I}}\mathcal{A}(I)$*[2].

Notice that the *dual precosheaf*

$$\hat{\mathcal{A}}(I) = \mathcal{A}(I')'$$

[1] As $\mathcal{I}$ is not an inductive family the terminology "precosheaf" is more appropriate, but the name "net" has acquainted more familiarity.

[2] If the precosheaf $\mathcal{A}$ is local, i.e. the algebras associated with disjoint intervals commute (see below), last property follows from the cyclicity property.

is a conformal net on S^1 with the same representation U and, since $\vee\hat{\mathcal{A}}(I) = \cap\mathcal{A}(I)$, Ω is cyclic for $\hat{\mathcal{A}}$. Of course

$$\hat{\hat{\mathcal{A}}} = \hat{\mathcal{A}} \ .$$

Let r be an orientation reversing isometry of S^1 with $r^2 = 1$ (e.g. r_{I_0}). The action of r on $PSL(2,\mathbb{R})$ by conjugation lifts to an action σ_r on G. We denote by Γ the semidirect product of G with $\mathbb{Z}_2$ via σ_r. Since Γ is a covering of the group generated by $PSL(2,\mathbb{R})$ and r, Γ acts on S^1. We call (anti-)unitary a representation U of Γ with operators on $\mathcal{H}$ such that $U(g)$ is unitary, resp. antiunitary, when g is orientation preserving, resp. orientation reversing.

The results in the following are known in the local case, but we stress their independence of any local commutativity assumption. The exposition follows in part [13]. Where there are variations, we give a proof.

Theorem 2.1. *Let $\mathcal{A}$ be a conformal precosheaf on S^1. Then the following properties hold:*

(i) Reeh-Schlieder theorem*: Ω is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$*

(ii) Modular covariance*: For any $I \in \mathcal{I}$ the modular group of $\mathcal{A}(I)$ wuth respect to Ω has the geometric meaning corresponding to Λ_I, namely*

$$\Delta_I^{it}\mathcal{A}(I_0)\Delta_I^{-it} = \mathcal{A}(\Lambda_I(-2\pi t)I_0), \ I_0 \in \mathcal{I}, \ t \in \mathbb{R},$$

where $\Delta_I = \Delta_{\mathcal{A}(I)}$ denotes the modular operator associated with $(\mathcal{A}(I), \Omega)$;

the one-parameter group $z(t) = z_{\mathcal{A}}(t)$ of unitaries defined by

$$U(\Lambda_I(-2\pi t)) = \Delta_I^{it} z_{\mathcal{A}}(t)$$

commutes with $U(g)$, $g \in G$, and belongs to the center of the gauge group[3].

In particular the unitary, positive energy, representation U of G is uniquely determined by $\mathcal{A}$ by the formula

$$U(\Lambda_I(2\pi t)) = \Delta_I^{-i\frac{t}{2}} \Delta_{I'}^{i\frac{t}{2}} \ .$$

Moreover U extends to an (anti-)unitary representation of Γ determined by

$$U(r_I) = J_I, \ I \in \mathcal{I}, \tag{2.1}$$

where J_I is the modular conjugation associated with $(\mathcal{A}(I), \Omega)$. If $g \in \Gamma$ is orientation reversing, then

$$U(g)\mathcal{A}(I)U(g)^* = \hat{\mathcal{A}}(gI) \ . \tag{2.2}$$

$\mathcal{A}$ and $\hat{\mathcal{A}}$ have the same unitary representation of G and

$$\Delta_{\hat{\mathcal{A}}(I)}^{it} = \Delta_{\mathcal{A}(I)}^{it} z_{\mathcal{A}}(2t) \ . \tag{2.3}$$

[3]We mean here the group of al unitaries V on $\mathcal{H}$ such that $V\Omega = \Omega$ and $V\mathcal{A}(I)V^* = \mathcal{A}(I)$, $I \subset \mathcal{I}$.

(iii) Additivity (and continuity) *: if I, I_k are intervals, and $I \subset \cup_k I_k$, then*

$$\mathcal{A}(I) \subset \vee_k \mathcal{A}(I_k);$$

if $\bar{I}$ denotes the closure of $I \in \mathfrak{I}$ and $\bar{I} \supset \cap_k I_k$ then, then

$$\mathcal{A}(\bar{I}) \supset \cap_k \mathcal{A}(I_k),$$

in particular $\mathcal{A}(I) = \mathcal{A}(\bar{I})$.

(iv) $U(2\pi) = \pm 1$

Proof. The proof of (i) and (iii) are as in the local case, see [11].

(ii): First we observe that, for any $g \in G$, $I \in \mathfrak{I}$,

$$\begin{aligned}\Delta_I^{it} U(g) \Delta_I^{-it} &= U(\Lambda_I(-2\pi t) g \Lambda_I(2\pi t)) \\ J_I U(g) J_I &= U(r_I g r_I) \ .\end{aligned}$$

These relations hold for $g = T_I(s)$ and for $g = T_{I'}(s)$ by Borchers' theorem [5], hence for any g because T_I and $T_{I'}$ generate G. Then, if $I_1 \in \mathfrak{I}$, we may find $g \in G$ such that $I_1 = gI$, therefore

$$\begin{aligned}\Delta_I^{it} \mathcal{A}(I_1) \Delta_I^{-it} &= \Delta_I^{it} \mathcal{A}(gI) \Delta_I^{-it} \\ &= \Delta_I^{it} U(g) \Delta_I^{-it} \mathcal{A}(I) \Delta_I^{it} U(g^{-1}) \Delta_I^{-it} \\ &= U(\Lambda_I(-2\pi t) g \Lambda_I(2\pi t)) \mathcal{A}(I) U(\Lambda_I(-2\pi t) g^{-1} \Lambda_I(2\pi t)) \\ &= \mathcal{A}(\Lambda_I(-2\pi t) g \Lambda_I(2\pi t) I) = \mathcal{A}(\Lambda_I(-2\pi t) I_1) \ .\end{aligned}$$

For any $t \in \mathbb{R}$, $z(t) = \Delta_I^{it} U(\Lambda_I(2\pi t))$ is thus an automorphism of any local algebra, hence commutes with Δ_I^{is}, which implies that $t \to z(t)$ is a one parameter group, commutes with $U(g)$, $g \in \Gamma$, and is independent of I. Due to the same reason $z(t)$ commutes with J_I. Then, setting $U(r_I) = J_I$, $g \in \Gamma \to U(g)$ is an (anti-)unitary representation which commutes with $z(t)$.

Finally, if g is orientation reversing, then $g r_I$ is orientation preserving, therefore

$$\begin{aligned}\mathrm{Ad}U(g)\mathcal{A}(I) = \mathrm{Ad}U(g r_I r_I)\mathcal{A}(I) &= \mathrm{Ad}U(g r_I)\mathrm{Ad}J_I \mathcal{A}(I) \\ &= \mathrm{Ad}U(g r_I)\hat{\mathcal{A}}(I') = \hat{\mathcal{A}}(g r_I I') = \hat{\mathcal{A}}(gI) \ .\end{aligned}$$

By uniqueness, the representation U of $\mathcal{A}$ is also the representation associated with $\hat{\mathcal{A}}$.

To show (2.3), notice first that $z_{\mathcal{A}}(t)$ commutes with J_I, as it implements an automorphism of $\mathcal{A}(I)$, hence with $U(g)$, $g \in \Gamma$ because of its definition in (2.1). Let then $g \in \Gamma$ be orientation reversing with $gI = I$, thus $g \Lambda_I(t) g^{-1} = \Lambda_I(-t)$. We have

$$\begin{aligned}\Delta_{\hat{\mathcal{A}}(I)}^{it} = U(g) \Delta_{\mathcal{A}(I)}^{-it} U(g)^* &= U(g) U(\Lambda(2\pi t)) z_{\mathcal{A}}(t) U(g)^* \\ &= U(\Lambda(-2\pi t)) z_{\mathcal{A}}(t) = \Delta_{\mathcal{A}(I)}^{it} z_{\mathcal{A}}(2t).\end{aligned}$$

The above formula also entails that $z(t)$ commutes with every unitary V in the gauge group, as such a V commutes both with $\Delta^{it}_{\mathcal{A}(I)}$ and $\Delta^{it}_{\hat{\mathcal{A}}(I)}$ by the modular theory.

(iv): Since $U(2\pi)$ is an automorphism of $\mathcal{A}(I)$, it commutes with the associated modular antiunitary J_I. Then,

$$U(2\pi) = J_I U(2\pi) J_I = U(r_I R(2\pi) r_I) = U(-2\pi) \ .$$

□

Remark. The group G acts on on the n-covering and on the universal covering of S^1 (homeomorphic to $\mathbb{R}$) and one may consider more general precosheaves on these spaces. There are direct extensions of the above results in these cases, in particular concerning modular covariance. We omit this generalization for simplicity.

We shall say that $\mathcal{A}$ satisfies *twisted locality* if there exists a unitary Z, commuting with the unitary representation U and with $z_{\mathcal{A}}(t)$, such that $Z\Omega = \Omega$ and

$$Z\mathcal{A}(I')Z^* \subset \mathcal{A}(I)'$$

for all intervals I.

Proposition 2.2. *Let $\mathcal{A}$ satisfy twisted locality. Then*

(i) Twisted duality *holds:*

$$Z\mathcal{A}(I')Z^* = \mathcal{A}(I)', \quad I \in \mathcal{I} \ .$$

i.e. $\hat{\mathcal{A}}(I) = Z\mathcal{A}(I)Z^*, \ I \in \mathcal{I}$,

(ii) The Bisognano-Wichmann property *holds:*

$$\Delta^{it}_I = U(\Lambda_I(-2\pi t)),$$

namely $z(t) = 1$.

Proof. (i): By twisted locality $Z\mathcal{A}(I)Z^* \subset \hat{\mathcal{A}}(I)$, moreover Ω is cyclic and separating for both $\hat{\mathcal{A}}(I)$ and $Z\mathcal{A}(I)Z^*$. Now the modular group $\mathrm{Ad}\Delta^{it}_{\hat{\mathcal{A}}(I)}$ of $\hat{\mathcal{A}}(I)$ leaves $Z\mathcal{A}(I)Z^*$ globally invariant by equation (2.3) and the commutativity between Z and U,$z(t)$; hence $\hat{\mathcal{A}}(I) = Z\mathcal{A}(I)Z^*$ by Takesaki's theorem.

(ii): We have $z(t) = \Delta^{it}_I U(\Lambda_I(2\pi t))$ independently of the interval I, hence $z(t) = \Delta^{it}_{I'} U(\Lambda_{I'}(2\pi t)) = \Delta^{-it}_I U(\Lambda_I(-2\pi t)) = z(-t)$, thus $z(t) = 1$, where we have used the twisted duality property to entail that $\Delta^{-it}_{I'} = Z\Delta^{it}_I Z^* = \Delta^{it}_I$, as Z commutes with $\Delta^{it}_{I'}$. □

Since $U(2\pi)$ is an involutive automorphism of any local algebra and has square 1, we may define the Fermi and Bose part of $\mathcal{A}(I)$ as

$$\mathcal{A}_\pm(I) := \{x \in \mathcal{A}(I) : U(2\pi)xU(2\pi) = \pm x\} \ .$$

As is known and easy to check, normal commutation relations are equivalent to twisted locality with the unitary Z given by

$$Z = \frac{I + iU(2\pi)}{1+i} \ . \tag{2.4}$$

In the following proposition we show that a weak form of twisted locality holds, i.e. the vacuum expectations of the commutators vanish.

Proposition 2.3. *Let $\mathcal{A}$ a conformal precosheaf on S^1. Then* weak twisted locality *holds i.e., for any $I \in \mathcal{I}$,*

$$([x, ZyZ^*]\Omega, \Omega) = 0 \ , \qquad x = x^* \in \mathcal{A}(I), \ y = y^* \in \mathcal{A}(I'),$$

where Z is given by formula (2.4). In particular weak locality *(i.e. $([x, y]\Omega, \Omega) = 0$) is equivalent to $U(2\pi) = 1$*

Proof. By conformal invariance, it is sufficient to prove the weak twisted locality for the upper semicircle I_0. This amounts to show that when x is a selfadjoint element in $\mathcal{A}(I_0)$ and y is a selfadjoint element in $\mathcal{A}(I_0')$ then $(x\Omega, Zy\Omega)$ is real.

Since Z commutes with $U(g)$ when g is orientation preserving and $JZJ = Z^*$, and making use of the commutation relations following from Theorem 2.1, a straightforward computation shows that, if S_0 is the Tomita operator for $(\mathcal{A}(I_0), \Omega)$, then $ZU(\pi)S_0U(-\pi) = S_0^* Z$. Therefore we get

$$\begin{aligned} (x\Omega, Zy\Omega) &= (x\Omega, ZU(\pi)S_0U(-\pi)y\Omega) = (x\Omega, S_0^* Zy\Omega) &(2.5)\\ &= (Zy\Omega, x\Omega) = \overline{(x\Omega, Zy\Omega)} \ . &(2.6) \end{aligned}$$

Finally let us assume weak locality and suppose by contradiction that $U(2\pi) \neq 1$. Then there exists $I \in \mathcal{I}$ and a non-zero selfadjoint $x \in \mathcal{A}(I)$ such that $U(2\pi)x\Omega = -x$, hence, for any $y = y^* \in \mathcal{A}(I')$, $(x\Omega, y\Omega)$ is real by weak locality and $(Zx\Omega, y\Omega) = i(x\Omega, y\Omega)$ is real too, namely $(x\Omega, y\Omega) = 0$ for any $y \in \mathcal{A}(I')$, and this implies the thesis by the Reeh-Schlieder property. □

We shall say that $\mathcal{A}$ is *irreducible* if the von Neumann algebra $\vee\mathcal{A}(I)$ generated by all local algebras coincides with $\mathcal{B}(\mathcal{H})$. The irreducibility property is indeed equivalent to several other requirement.

Proposition 2.4. *Assume $z_{\mathcal{A}}(t) = 1$. The following are equivalent:*

(i) $\mathbb{C}\Omega$ are the only $U(G)$ invariant vectors.

(ii) The algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, are factors. In particular they are type III_1 factors, provided $\mathcal{A} \neq \mathbb{C}$.

(iii) The net $\mathcal{A}$ is irreducible.

(iv) The dual net $\hat{\mathcal{A}}$ is irreducible, i.e. the algebra $\cap\mathcal{A}(I)$ given by the intersection of all local algebras coincides with $\mathbb{C}$.

Proof. The proof is similar to the one given in [13] in the local case; one just notices that property (i) is the same for $\mathcal{A}$ and $\hat{\mathcal{A}}$, thus (iii) and (iv) are equivalent. $\square$

Irreducibility is also equivalent to Ω being unique invariant for any of the unitary subgroups corresponding to T_I, Λ_I or R, see Lemma B.2 of the appendix [13].

In the next Corollary the assumption of compactness $\{\mathrm{Ad}z_{\mathcal{A}}(t),\ t \in \mathbb{R}\}^-$ is satisfied in particular if $\mathcal{A}$ is distal split [10] (see below) or, of course, if $\mathcal{A}$ is twisted local.

Corollary 2.5. *If $\mathcal{A} \neq \mathbb{C}$ is irreducible and $\{\mathrm{Ad}z_{\mathcal{A}}(t),\ t \in \mathbb{R}\}^-$ is compact, then $\mathcal{A}(I),\ I \in \mathcal{I}$, is a type III factor. If $z(t) = 1$, then $\mathcal{A}(I)$ is of type III_1.*

Proof. By assumption the closure (in the gauge automorphism group) $\mathcal{G}_0$ of $\{\mathrm{Ad}z_{\mathcal{A}}(t),$ $t \in \mathbb{R}\}$ is a compact and abelian. Denote by $\mathcal{A}^z$ the fixed-point net under the action of $\mathcal{G}_0$. If $\mathcal{A}^z(I) = \mathbb{C}$, namely $\mathcal{G}_0$ acts ergodically, then $\mathcal{A}(I)$ must be abelian, hence $\mathcal{A}$ is local and $z_{\mathcal{A}}(t) = 1$ (Prop. 2.7), but this implies that $\mathcal{A}^z = \mathcal{A}$ is trivial.

So we may assume that $\mathcal{A}^z$ is non-trivial, hence $\mathcal{A}^z(I)$ is a type III_1 factor by the uniqueness of the vacuum and Proposition 2.4. By a similar reasoning the relative commutant $\mathcal{R}(I) = \mathcal{A}^z(I)' \cap \mathcal{A}(I)$ is abelian as $\mathcal{G}_0$ acts ergodically on it, hence $\mathcal{R}$ has to be a constant local precosheaf. Then $U(\Lambda_I(-2\pi t))$ and $z_{\mathcal{A}}(t)$ have the same restriction to the Hilbert space of $\mathcal{R}$, thus $\mathcal{G}_0$ would act trivially on $\mathcal{R}$ (because by lemma 3.4 $U(\Lambda_I(-2\pi t))$ has no eigenvalue other than 1, while $z(t)$ has pure point spectrum) and $\mathcal{R}(I) = \mathbb{C}$. It follows that $\mathcal{A}(I)$ is a factor that must be of type III because it has a normal conditional expectation onto $\mathcal{A}^z(I)$. $\square$

We include the following corollary, a variant of a result in [4].

Corollary 2.6. *Let $\mathcal{A}$ be an irreducible conformal net with $\{z_{\mathcal{A}}(t),\ t \in \mathbb{R}\}^-$ compact and let $\mathcal{A}_0$ be its restriction to $\mathbb{R} = S^1\backslash\{-1\}$. Let $\mathcal{B}_0 \subset \mathcal{A}_0$ be a subnet on $\mathbb{R}$, with the same translations and dilations of $\mathcal{A}_0$, Then $\mathcal{B}_0(\mathbb{R}^+)' \cap \mathcal{A}_0(\mathbb{R}^+)$ is either trivial or a type III factor. If moreover $\mathcal{B}_0 \subset \mathcal{A}_0$ has finite index, i.e. $[\mathcal{B}_0(\mathbb{R}^+) : \mathcal{A}_0(\mathbb{R}^+)] < \infty$, then the first case occurs: $\mathcal{B}_0(\mathbb{R}^+)' \cap \mathcal{A}_0(\mathbb{R}^+) = \mathbb{C}$.*

Proof. We assume that $\mathcal{A}$ is local; the general case follows by similar arguments. As the translations and dilations of $\mathcal{A}_0$ restrict to $\mathcal{B}_0$, it follows that $\mathcal{B}_0$ extends to a conformal precosheaf on S^1 [14]. Moreover the uniqueness of the vacuum holds for $\mathcal{A}_0$ hence for $\mathcal{B}_0$ too and therefore $\mathcal{B}(\mathbb{R}^+)$ is type III unless $\mathcal{A}$ is trivial (Prop. 2.5). By the same reason $\mathcal{B}_0(\mathbb{R}^+)' \cap \mathcal{A}_0(\mathbb{R}^+)$ is either trivial or a type III factor, but the last possibility cannot occur in the finite-index case because $\mathcal{A}_0(\mathbb{R}^+)' \cap \mathcal{A}(\mathbb{R}^+)$ is then finite-dimensional. $\square$

We shall say that $\mathcal{A}$ is *local* if whenever I_1, I_2 are disjoint intervals the two algebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute.

Remark. The "observable net" $I \to \mathcal{C}(I) = \mathcal{A}(I) \cap \hat{\mathcal{A}}(I)$ is local and, by modular covariance and Takesaki's theorem, for each I there exists a normal, vacuum-preserving, conditional expectation ε_I from $\mathcal{A}(I)$ (or from $\hat{\mathcal{A}}(I)$ onto $\mathcal{C}(I)$) and $\varepsilon_{\tilde{I}}|_{\mathcal{A}(I)} = \varepsilon_I$ if $I \subset \tilde{I}$.

Proposition 2.7. *Let $\mathcal{A}$ be a conformal precosheaf on S^1. The following are equivalent:*

(i) *$\mathcal{A}$ is local.*

(ii) *Haag duality holds, i.e. $\mathcal{A}(I') = \mathcal{A}(I)'$, $I \in \mathcal{I}$.*

(iii) *The vacuum is cyclic for the algebra $\mathcal{A}(I) \cap \hat{\mathcal{A}}(I)$ for some (hence for any) $I \in \mathcal{I}$.*

(iv) *The algebras $\mathcal{A}(I)$ and $\hat{\mathcal{A}}(I)$ coincide for some (hence for any) $I \in \mathcal{I}$.*

If these properties hold then U is indeed a representation of $PSL(2,\mathbb{R})$, i.e. $U(2\pi) = 1$.

Proof. The relations *(ii)* $\Rightarrow$ *(i)*, *(iv)* $\Rightarrow$ *(iii)*, *(iv)* $\Leftrightarrow$ *(ii)* are obvious, and the implication *(i)* $\Rightarrow$ *(ii)* is proven e.g. in [6]. If *(iii)* holds, then $\mathcal{A}(I) \cap \hat{\mathcal{A}}(I)$ is a subalgebra of $\mathcal{A}(I)$ for which Ω is cyclic. Since $\Delta^{it}_{\hat{\mathcal{A}}(I)} = \Delta_{\mathcal{A}(I)} z(2t)$, we have that $\mathcal{A}(I) \cap \hat{\mathcal{A}}(I)$ is stable under the modular group of $\hat{\mathcal{A}}(I)$ with respect to Ω, hence coincides with $\hat{\mathcal{A}}(I)$. By the same argument it coincides with $\hat{\mathcal{A}}(I)$, too. Finally if the net is local it is in particular weakly local, and this, according to Proposition 1.2, implies the thesis. $\square$

2.1 Wiesbrock's theorem in the non-local case

In order to clarify the general structure, we give here the necessary modifications in order to characterize conformal nets on S^1, also in the non-local case, by a simple extension of Wiesbrock's theorem with *half-sided modular inclusions* (hsm). This is however independent of the rest of the paper.

Recall that $(\mathcal{R} \subset \mathcal{S}, \Omega)$ is a $\pm$hsm if $\mathcal{R} \subset \mathcal{S}$ are von Neumann algebra, Ω is a cyclic and separating vector for both $\mathcal{R}$ and $\mathcal{S}$ and $\mathrm{Ad}\Delta^{it}_{\mathcal{S}}\mathcal{S} \subset \mathcal{S}$, $\pm t > 0$, where $\Delta_{\mathcal{S}}$ is the modular operator associated with $(\mathcal{S}, \Omega)$.

Let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras on a Hilbert space $\mathcal{H}$ and Ω a cyclic and separating vector for $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}$. Consider the following properties:

(i) $(\mathcal{N} \cap \mathcal{M} \subset \mathcal{M}, \Omega)$ is -hsm,

(ii) $(\mathcal{N}' \cap \mathcal{M} \subset \mathcal{M}, \Omega)$ is +hsm,

(iii) $J_{\mathcal{N}} J_{\mathcal{M}} = \pm J_{\mathcal{M}} J_{\mathcal{N}}$, and $J_{\mathcal{N}} \Delta^{it}_{\mathcal{M}} = \Delta^{-it}_{\mathcal{M}} J_{\mathcal{N}}$

(iii') $J_{\mathcal{N}} J_{\mathcal{M}} = \pm J_{\mathcal{M}} J_{\mathcal{N}}$ and the unitaries $z(t)$ defined by $J_{\mathcal{N}} \Delta^{it}_{\mathcal{M}} = z(2t) \Delta^{-it}_{\mathcal{M}} J_{\mathcal{N}}$ satisfy $\mathrm{Ad}z(t)\mathcal{M} = \mathcal{M}$, $\mathrm{Ad}z(t)\mathcal{N} = \mathcal{N}$, $t \in \mathbb{R}$.

Property *(iii)* holds in particular if $J_{\mathcal{N}} \mathcal{M} J_{\mathcal{N}} = \mathcal{M}$ and this, together with properties *(i)* and *(ii)* characterizes *local* conformal nets [21, 1]. The following variation holds.

Theorem 2.8. *Let $\mathcal{N}$ and $\mathcal{M}$ and Ω satisfy properties (i), (ii) and (iii') above. There exists a unique conformal net $\mathcal{A}$ on S^1 such that $\mathcal{A}(I_1) = \mathcal{M}$, $\mathcal{A}(I_2) = \mathcal{N}$, with I_1 and I_2 the upper and right semicircles, and Ω is the vacuum vector. Moreover $z_{\mathcal{A}}(t) = z(t)$.*

All conformal nets on S^1 arise in this way.

Proof. For simplicity we assume $z(t) = 1$, the general case can be treated similarly.

Set $\hat{\mathcal{M}} = J_{\mathcal{N}}\mathcal{M}J_{\mathcal{N}}$ so that, by property (iii)

$$\Delta^{it}_{\hat{\mathcal{M}}} = \Delta^{it}_{\mathcal{M}}, \quad J_{\hat{\mathcal{M}}} = \pm J_{\mathcal{M}} \ ,$$

and notice that:

(a) $(\mathcal{N} \cap \mathcal{M} \subset \mathcal{M}, \Omega)$ is -hsm ,

(b) $(\mathcal{N}' \cap \hat{\mathcal{M}} \subset \hat{\mathcal{M}}, \Omega)$ is +hsm ,

(c) $(\mathcal{N}' \cap \hat{\mathcal{M}} \subset \mathcal{N}' \vee \mathcal{M}', \Omega)$ is -hsm.

These are analogous to the corresponding properties in the proof of Theorem 3 of [21]. (a) is just (i), (b) follows from (a) by applying $\mathrm{Ad}J_{\mathcal{N}}$, and (c) follows by some elementary modification of Wiesbrock proof in [21] where, from formula (5) up to formula (8), one replaces $\mathcal{N}' \cap \mathcal{M}$ with $\mathcal{N}' \cap \hat{\mathcal{M}}$.

It follows that the modular unitary groups $\Delta^{it_1}_{\mathcal{N}\cap\mathcal{M}}$, $\Delta^{it_2}_{\mathcal{N}'\cap\hat{\mathcal{M}}}$ and $\Delta^{it_3}_{\mathcal{M}}$ mutually have the same commutation relations as the one-parameter subgroups Λ_{I_1}, Λ_{I_2} and Λ_{I_3} of $PSL(2,\mathbb{R})$, with I_1 the upper-right quarter-circle, I_2 the lower-right quarter-circle, and I_3 the right half-circle. Therefore the unitary group generated by these three modular unitary groups provide a representation U of the universal cover G of $PSL(2,\mathbb{R})$, by an argument analogous to the one given in the proof of [14], Theorem 1.2.

Again analogously to [21], formula (12), we can see that $U(\pi) = J_{\mathcal{N}}J_{\mathcal{M}}$, hence

$$U(2\pi) = J_{\mathcal{N}}J_{\mathcal{M}}J_{\mathcal{N}}J_{\mathcal{M}} = \pm 1 \ ,$$

namely U is a representation of $SL(2,\mathbb{R})$.

Set $\mathcal{A}(I_0) = \mathcal{N}$, with I_0 the upper semi-circle and $\mathcal{A}(I) = \mathcal{A}(gI_0)$ if $I \in \mathcal{I}$ and $g \in G$ satisfy $I = gI_0$. Then $\mathcal{A}(I)$ is well-defined and isotonous as in the proof of [14], Theorem 1.2. The rest follows by standard arguments.

That all conformal nets arise in this way follows by Theorem 2.1. □

Remark. By relaxing the condition $J_{\mathcal{M}}J_{\mathcal{N}} = \pm J_{\mathcal{N}}J_{\mathcal{M}}$ to $J_{\mathcal{M}}J_{\mathcal{N}} = \mu J_{\mathcal{N}}J_{\mathcal{M}}$ for some $\mu \in \mathbb{T}$, one obtains a characterization of conformal nets on covers of S^1.

3 Maximal temperature and the split property. Examples from free probability

Recall now that a net $\mathcal{A}$ is said to satisfy the *split property* if there exists an intermediate type I factor $\mathcal{A}(I_1) \subset F \subset \mathcal{A}(I_2)$ whenever the closure of the interval I_1 is contained in the interior of the interval I_2 [10].

A weak form of this is the *distal split* property stating that for each $I \in \mathcal{I}$ there exists a $\tilde{I} \supset I$ and a type I factor F such that $\mathcal{A}(I) \subset F \subset \mathcal{A}(\tilde{I})$.

The following Proposition may be traced back to old argument of Kadison and has been used in [17, 7].

Proposition 3.1. *If the split property holds, then the local algebras $\mathcal{A}(I)$ are approximately finite-dimensional.*

Proof. By continuity we may suppose that I is open. Let $I_1 \subset I_2 \subset \cdots \subset I$ be an increasing sequence of open intervals with $\bar{I}_k \subset I_{k+1}$ and $\cup I_k = I$ and choose type I factors $\mathcal{A}(I_k) \subset F_k \subset \mathcal{A}(I_{k+1})$. Then $\mathcal{A}(I)$ is generated by the increasing sequence of type I factors F_k, hence it is approximately finite-dimensional. □

Let $\mathcal{A}$ be a conformal net and L be its conformal Hamiltonian. We shall say that $\mathcal{A}$ satisfies the *trace class condition* if

$$\mathrm{Tr}(e^{-\beta L}) < \infty \ , \forall \beta > 0.$$

Theorem 3.2. *If $\mathcal{A}$ satisfies the trace class condition, then $\mathcal{A}$ is split.*

Proof. By [8], Proposition 4.1, it is sufficient to show that $e^{-\beta L}$ has order 0 for all $\beta > 0$ and this follows if $e^{-\beta L}$ is of type l^p for all $p > 0$, [8] Lemma 2.1. The conclusion is thus a consequence of the following elementary lemma. □

Lemma 3.3. *If $\mathrm{Tr}(e^{-\beta L}) < \infty \ , \forall \beta > 0$, then, for any fixed $\beta > 0$, $e^{-\beta L}$ is of type l^p for all $p > 0$.*

Proof. Let ν_n be the multiplicity of the eigenvalue $2\pi n$ of L. Then the p-norm of $e^{-\beta L}$ is

$$||e^{-\beta L}||_p = (\sum_n e^{-\beta 2\pi n p}\nu_n)^{\frac{1}{p}} = \mathrm{Tr}(e^{-\beta p L})^{\frac{1}{p}}$$

therefore the trace class condition implies $||e^{-\beta L}||_p < \infty$ for all $p > 0$. □

Remark. By the Kohlbecker's Tauberian theorem [2] the trace class condition $\mathrm{Tr}(e^{-\beta L}) < \infty$, $\forall \beta > 0$, sets bounds for the growth of $\mathrm{Tr}(e^{-\beta L})$ as $\beta \to 0^+$. We now recall the following Lemma that we will need here below.

Lemma 3.4. *Let V be a non-trivial positive energy irreducible unitary representation of $PSL(2,\mathbb{R})$. Then*

(i) The restriction of V to the upper triangular (“$ax+b$”) group is irreducible.

(ii) The one-parameter unitary group $V\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is unitarily equivalent to the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$.

Proof. (i): See e.g. [14], comments after Theorem 2.1. Concerning (ii) we recall that, as the logarithm of the generator of the translation unitary group and the generator of the dilation unitary group satisfies the canonical commutation relation and are jointly irreducible, the result follows by von Neumann uniqueness theorem. □

3.1 Boltzmann statistics and maximal temperature

We now apply results in the previous sections to the construction of a non-split net that satisfies a trace class condition with maximal temperature.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F}(\mathcal{H})$ the Fock space over $\mathcal{H}$ with Boltzmann statistics

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k \ ,$$

where $\mathcal{H}_0 = \mathbb{C}\Omega$, Ω the vacuum vector, and $\mathcal{H}_k$ is the k-fold tensor product $\mathcal{H} \otimes \mathcal{H} \cdots \otimes \mathcal{H}$.

Let $\ell(h)$ and $r(k)$, $h, k \in \mathcal{H}$ be the left and right creation operator

$$\begin{aligned} \ell(h)|_{\mathcal{H}_k} &: \varphi_1 \otimes \cdots \otimes \varphi_k \to h \otimes \varphi_1 \otimes \cdots \otimes \varphi_k \\ r(h)|_{\mathcal{H}_k} &: \varphi_1 \otimes \cdots \otimes \varphi_k \to \varphi_1 \otimes \cdots \otimes \varphi_k \otimes h \end{aligned}$$

and

$$s(h) = \ell(h) + \ell(h)^*, \quad d(h) = r(h) + r(h)^*$$

the right and left fields. We have the following commutation relations:

$$\begin{aligned} [\ell(h), r(k)] &= 0 \\ [\ell(h)^*, r(k)^*] &= 0 \\ [\ell(h), r(k)^*] &= (h,k)P_\Omega \\ [\ell(h)^*, r(k)] &= (k,h)P_\Omega \end{aligned}$$

with P_Ω the one-dimensional projection onto $\mathbb{C}\Omega$, hence

$$[s(h), d(k)] = 2i\mathrm{Im}(h,k)P_\Omega \ . \tag{3.1}$$

If $H \subset \mathcal{H}$ is a real Hilbert space we define the von Neumann algebras

$$\mathcal{A}(H) = \{s(h), h \in H\}'', \quad \mathcal{B}(H) = \{d(h), h \in H\}''.$$

Note that H is standard, i.e. $\overline{H + iH} = \mathcal{H}$ and $H \cap iH = \{0\}$, iff Ω is cyclic and separating for $\mathcal{A}(H)$ or, equivalently, for $\mathcal{B}(H)$.

With Z the unitary involution

$$Z|_{H_n} : \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n \to \varphi_n \otimes \varphi_{n-1} \otimes \cdots \otimes \varphi_1$$

we have $Z\ell(h)Z = r(h)$, hence

$$Z\mathcal{A}(H)Z = \mathcal{B}(H).$$

Proposition 3.5 ([19]). *If H is standard, then*

(i) $\mathcal{A}(H)' = \mathcal{B}(H') = Z\mathcal{A}(H')Z$ where $H' = \{k \in \mathcal{H}, \mathrm{Im}(h,k) = 0 \ \forall h \in H\}$ is the symplectic complement of H

(ii) If the modular unitary group δ_H^{it} of H on $\mathcal{H}$ is unitarily equivalent to the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$, then $\mathcal{A}(H)$ is a type III_1-factor with core isomorphic to $L(\mathbb{F}_\infty) \otimes B(\ell^2(\mathbb{N}))$.[4]

We now specialize $\mathcal{H}$ to be the one-particle Hilbert space $\mathcal{H}^{(n)}$ associated with the $(n-1)^{th}$-derivative of the $U(1)$-current algebra and let $H^{(n)}(I)$ be the corresponding real standard Hilbert subspace generated by the smooth functions with support in the interval I of S^1, see [14]. Set

$$\mathcal{A}_n(I) = \mathcal{A}(H^{(n)}(I)), \quad \mathcal{B}_n(I) = \mathcal{B}(H^{(n)}(I)).$$

By the above proposition, the nets $\mathcal{A}_n$ are twisted local.

Corollary 3.6. *$\mathcal{A}_n(I)$ is a type III_1-factor with core isomorphic to $L(\mathbb{F}_\infty) \otimes B(\ell^2(\mathbb{N}))$.*

Proof. By Proposition 2.4 the type III_1 factor property by the irreduciblity of the net which holds because $H(I)$ is standard and $\{s(h),\ h \in \mathcal{H}\}'' = B(\mathcal{H})$.

Concerning the isomorphism class of $\mathcal{A}_n(I)$, by Proposition 3.5 it is sufficient to show that $\delta^{it}_{H^{(n)}(I)}$ is isomorphic to the left regular representation of $\mathbb{R}$. Now $\delta^{it}_{H^{(n)}(I)}$ is unitarily equivalent to $u(\Lambda(-2\pi t))$, with u the positive energy irreducible representation of $PSL(2,\mathbb{R})$ on $\mathcal{H}$. But this follow by Lemma 3.4. $\square$

Proposition 3.7.

(i) $\mathcal{B}_n(I) = \hat{\mathcal{A}}_n(I)$

(ii) $\mathcal{A}_n(I) \cap \mathcal{B}_n(I) = \mathbb{C}$

(iii) $\mathcal{A}_n(I) \vee \mathcal{B}_n(I) = B(\mathcal{H})$

Proof. (i) is immediate and therefore $(ii) \Leftrightarrow (iii)$. To show (iii) notice that $\mathcal{A}_n(I) \vee \mathcal{B}_n(I)$ contains P_Ω by 3.1, hence must be equal to $B(\mathcal{H})$ because Ω is cyclic. $\square$

We now let $l^{(n)}$ be the the generator of the rotation one-parameter unitary group on $\mathcal{H}^{(n)}$, so that the conformal Hamiltonian $L^{(n)}$ is the promotion of $l^{(n)}$ to $\mathcal{F}(\mathcal{H})$, $e^{itL^{(n)}}|_{\mathcal{H}_k} = e^{itl^{(n)}} \otimes e^{itl^{(n)}} \cdots \otimes e^{itl^{(n)}}$.

Proposition 3.8. *$\mathcal{A}_1$ satisfies the trace class condition with* maximal temperature *$\beta_1^{-1} = \frac{2\pi}{\log 2}$, namely*

$$\mathrm{Tr}(e^{-\beta L^{(1)}}) < +\infty, \quad \beta > \frac{\log 2}{2\pi} \tag{3.2}$$

$$\mathrm{Tr}(e^{-\beta L^{(1)}}) = +\infty, \quad \beta \leq \frac{\log 2}{2\pi} \tag{3.3}$$

[4]The core a type III_1 factor $\mathcal{L}$ is the crossed product of $\mathcal{L}$ by the action of $\mathbb{R}$ given by the modular group. $L(\mathbb{F}_\infty)$ is the von Neumann algebra generated by the left regular representation of the free group $\mathbb{F}_\infty$ on infinitely many generators. Here $\mathcal{A}(H)$ is isomorphic to the factor considered in [18].

Proof. Setting $L = L^{(1)}$, $l = l^{(1)}$ and $\mathcal{H} = \mathcal{H}^{(1)}$, the occurring unitary representation of $PSL(2,\mathbb{R})$ on $\mathcal{H}$ is the irreducible one with lowest weight 1, the spectrum of l is $\{2\pi n, n \in \mathbb{N}\}$ and each eigenvalue $2\pi n$ has multiplicity 1, hence

$$\mathrm{Tr}(e^{-\beta l}) = \sum_{k=1}^{\infty} e^{-\beta 2\pi k} = \frac{e^{-\beta 2\pi}}{1 - e^{-\beta 2\pi}}.$$

As $e^{-\beta L}|_{\mathcal{H}_n} = e^{-\beta l} \otimes \cdots \otimes e^{-\beta l}$, we have $\mathrm{Tr}(e^{-\beta L}|_{\mathcal{H}_n}) = \mathrm{Tr}(e^{-\beta l})^n$, hence we have

$$\mathrm{Tr}(e^{-\beta L}) = \sum_{k=0}^{\infty} \Big(\frac{e^{-\beta 2\pi}}{1 - e^{-\beta 2\pi}}\Big)^k < +\infty \Leftrightarrow \beta > \frac{\log 2}{2\pi}.$$

□

The above proposition can now be easily generalized in order to obtain the following.

Theorem 3.9. *For any $T > 0$, there exists a conformal net on S^1 satisfying twisted locality and the trace class condition with maximal temperature $\beta^{-1} > T$, but not satisfying the split property.*

Proof. The irreducible unitary representation of $PSL(2,\mathbb{R})$ on $\mathcal{H}^{(n)}$ is the positive energy one with lowest weight n [14]. We consider the associated conformal nets $\mathcal{A}_n$ of von Neumann algebras on the full Fock space over $\mathcal{H}^{(n)}$. We have seen that $\mathcal{A}_1$ satisfies the trace class condition with maximal temperature $\frac{2\pi}{\log 2}$.

A similar computation with the conformal Hamitonian $L^{(n)}$ of $\mathcal{A}_n$ shows that $\mathrm{Tr}(e^{-\beta L^{(n)}}) = \sum_{k=0}^{\infty}(\frac{e^{-\beta 2\pi n}}{1-e^{-\beta 2\pi}})^k)$ which is finite iff $\beta > \beta_n$ where β_n, the solution of $\frac{e^{-\beta 2\pi n}}{1-e^{-\beta 2\pi}} = 1$, satisfies $\lim_{n\to\infty} \beta_n = 0$.

On the other hand $\mathcal{A}_n$ does not satisfies the split property because by Proposition 3.5 the $\mathcal{A}_n(I)$ have cores isomorphic to $L(\mathbb{F}_n) \otimes B(\ell^2(\mathbb{N}))$ hence they are not approximately finite-dimensional. □

Remark. It can be shown that, for each n, $\mathcal{A}_n$ does not even satisfies the “distal split” property.

We conclude our paper by pointing out the following question.

Problem. Besides twisted locality, is there a notion related to free independence fulfilled by the nets $\mathcal{A}_n$?

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